

# On the Number of Partitions with Designated Summands

William Y.C. Chen<sup>1</sup>, Kathy Q. Ji<sup>2</sup>, Hai-Tao Jin<sup>3</sup> and Erin Y.Y. Shen<sup>4</sup>

Center for Combinatorics, LPMC-TJKLC  
Nankai University, Tianjin 300071, P.R. China

<sup>1</sup>chen@nankai.edu.cn, <sup>2</sup>ji@nankai.edu.cn, <sup>3</sup>jinh1006@mail.nankai.edu.cn,  
<sup>4</sup>shenyiyi@mail.nankai.edu.cn

**Abstract.** Andrews, Lewis and Lovejoy introduced the partition function  $PD(n)$  as the number of partitions of  $n$  with designated summands, where we assume that among parts with equal size, exactly one is designated. They proved that  $PD(3n+2)$  is divisible by 3. We obtain a Ramanujan type identity for the generating function of  $PD(3n+2)$  which implies the congruence of Andrews, Lewis and Lovejoy. For  $PD(3n)$ , Andrews, Lewis and Lovejoy showed that the generating function can be expressed as an infinite product of powers of  $(1-q^{2n+1})$  times a function  $F(q^2)$ . We find an explicit formula for  $F(q^2)$ , which leads to a formula for the generating function of  $PD(3n)$ . We also obtain a formula for the generating function of  $PD(3n+1)$ . Our proofs rely on Chan's identity on Ramanujan's cubic continued fraction and some identities on cubic theta functions. By introducing a rank for the partitions with designed summands, we give a combinatorial interpretation of the congruence of Andrews, Lewis and Lovejoy.

## 1 Introduction

Andrews, Lewis and Lovejoy [2] investigated the number of partitions with designated summands which are defined on ordinary partitions by designating exactly one part among parts with equal size. Let  $PD(n)$  denote the number of partitions of  $n$  with designated summands. For example, there are ten partitions of 4 with designated summands:

$$\begin{array}{ccccccccc} 4', & 3' + 1', & 2' + 2, & 2 + 2', & 2' + 1' + 1, \\ 2' + 1 + 1', & 1' + 1 + 1 + 1, & 1 + 1' + 1 + 1, & 1 + 1 + 1' + 1, & 1 + 1 + 1 + 1'. \end{array}$$

The notion of partitions with designated summands goes back to MacMahon [10]. He considered partitions with designated summands and with exactly  $k$  different sizes, see also Andrews and Rose [5]. Andrews, Lewis and Lovejoy [2] derived the following generating function of  $PD(n)$ .

**Theorem 1.1.** *We have*

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{(q^6; q^6)_{\infty}}{(q; q)_{\infty}(q^2; q^2)_{\infty}(q^3; q^3)_{\infty}}, \quad (1.1)$$

where  $(a; q)_\infty$  stands for the  $q$ -shifted factorial

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}), \quad |q| < 1.$$

By using modular forms and  $q$ -series identities, Andrews, Lewis and Lovejoy showed that the partition function  $PD(n)$  has many interesting divisibility properties. In particular, they obtained the following Ramanujan type congruence.

**Theorem 1.2.** ([2, Corollary 7]) *For  $n \geq 0$ , we have*

$$PD(3n + 2) \equiv 0 \pmod{3}. \quad (1.2)$$

In this paper, we obtain the following Ramanujan type identity for the generating function of  $PD(3n + 2)$  which implies the above congruence.

**Theorem 1.3.** *We have*

$$\sum_{n=0}^{\infty} PD(3n + 2)q^n = 3 \frac{(q^3; q^6)_\infty^3 (q^6; q^6)_\infty^6}{(q; q^2)_\infty^5 (q^2; q^2)_\infty^8}. \quad (1.3)$$

Andrews, Lewis and Lovejoy also obtained explicit formulas for the generating functions for  $PD(2n)$  and  $PD(2n + 1)$  by using Euler's algorithm for infinite products [1, P. 98] and Sturm's criterion [12]. As for  $PD(3n)$ , they showed that the generating function permits the following form.

**Theorem 1.4.** ([2, Theorem 23]) *Define  $c(n)$  uniquely by*

$$\sum_{n=0}^{\infty} PD(3n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-c(n)}, \quad (1.4)$$

*then for all positive  $n$ ,*

$$\begin{aligned} c(6n + 1) &= 5, \\ c(6n + 3) &= 2, \\ c(6n + 5) &= 5. \end{aligned}$$

Equivalently, the above theorem says that there exists a series  $F(q^2)$  such that

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{1}{(q; q^6)_\infty^5 (q^3; q^6)_\infty^2 (q^5; q^6)_\infty^5} \times F(q^2). \quad (1.5)$$

In this paper, we find an explicit formula for  $F(q^2)$ , that is,

$$F(q^2) = \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^4}{(q^2; q^2)_\infty^{10} (q^{12}; q^{12})_\infty^2} + 3q^2 \frac{(q^{12}; q^{12})_\infty^6}{(q^2; q^2)_\infty^6 (q^4; q^4)_\infty^2}, \quad (1.6)$$

which leads to the following generating function of  $PD(3n)$ .

**Theorem 1.5.** *We have*

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{1}{(q; q^6)_{\infty}^5 (q^3; q^6)_{\infty}^2 (q^5; q^6)_{\infty}^5} \times \left( \frac{(q^4; q^4)_{\infty}^6 (q^6; q^6)_{\infty}^4}{(q^2; q^2)_{\infty}^{10} (q^{12}; q^{12})_{\infty}^2} + 3q^2 \frac{(q^{12}; q^{12})_{\infty}^6}{(q^2; q^2)_{\infty}^6 (q^4; q^4)_{\infty}^2} \right). \quad (1.7)$$

In fact, we obtain explicit formulas for the 3-dissection of the generating function of  $PD(n)$ , which include the following generating function for  $PD(3n+1)$ .

**Theorem 1.6.** *We have*

$$\sum_{n=0}^{\infty} PD(3n+1)q^n = \frac{(q^3; q^6)_{\infty}^3 (q^6; q^6)_{\infty}^6}{(q; q^2)_{\infty}^5 (q^2; q^2)_{\infty}^8} \left( 4q \frac{(q; q^2)_{\infty}^2}{(q^3; q^6)_{\infty}^6} + \frac{(q^3; q^6)_{\infty}^3}{(q; q^2)_{\infty}} \right). \quad (1.8)$$

Our dissection formulas rely on the Chan's identity on Ramanujan's cubic continued fraction [8] and cubic theta functions [6, 9]. In Section 3, we shall give a combinatorial interpretation of the congruence  $PD(3n+2) \equiv 0 \pmod{3}$  by introducing a rank for the partitions with designed summands.

## 2 Proofs

In this section, we give proofs of the generating functions for  $PD(3n)$ ,  $PD(3n+1)$  and  $PD(3n+2)$  by employing Chan's identity on Ramanujan's cubic continued fraction. It should be noted that the generating function of  $PD(3n)$  derived this way does not directly imply a formula for  $F(q^2)$ . To compute  $F(q^2)$ , we shall make use of some identities on cubic theta functions.

Recall that Ramanujan's cubic continued fraction  $v(q)$  is given by

$$v(q) := \frac{q^{\frac{1}{3}}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \dots$$

It is known that

$$v(q) = q^{\frac{1}{3}} \frac{(q; q^2)_{\infty}}{(q^3; q^6)_{\infty}^3},$$

see Andrews and Berndt [3, P. 94]. The following identity is due to Chan and will be used in our derivation of the 3-dissection formulas.

**Theorem 2.1.** ([8, Eq. (13)]) *We have*

$$\frac{1}{(q; q)_{\infty} (q^2; q^2)_{\infty}} = \frac{(q^9; q^9)_{\infty}^3 (q^{18}; q^{18})_{\infty}^3}{(q^3; q^3)_{\infty}^4 (q^6; q^6)_{\infty}^4} \times \left\{ \left( \frac{1}{x^2(q^3)} - 2q^3 x(q^3) \right) + q \left( \frac{1}{x(q^3)} + 4q^3 x^2(q^3) \right) + 3q^2 \right\}, \quad (2.1)$$

where

$$x(q) = q^{-\frac{1}{3}}v(q) = \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty}.$$

*Proof of Theorems 1.3 and 1.6.* Multiplying both sides of (2.1) by

$$\frac{(q^6; q^6)_\infty}{(q^3; q^3)_\infty},$$

we find

$$\begin{aligned} \frac{(q^6; q^6)_\infty}{(q; q)_\infty (q^2; q^2)_\infty (q^3; q^3)_\infty} &= \frac{(q^9; q^9)_\infty^3 (q^{18}; q^{18})_\infty^3}{(q^3; q^3)_\infty^5 (q^6; q^6)_\infty^3} \left\{ \left( \frac{1}{x^2(q^3)} - 2q^3 x(q^3) \right) \right. \\ &\quad \left. + q \left( \frac{1}{x(q^3)} + 4q^3 x^2(q^3) \right) + 3q^2 \right\}. \end{aligned} \quad (2.2)$$

Observe that the left-hand side of (2.2) is the generating function for  $PD(n)$ . Extracting those terms involving the powers  $q^{3n}$ ,  $q^{3n+1}$  and  $q^{3n+2}$ , respectively, we deduce that

$$\sum_{n=0}^{\infty} PD(3n) q^{3n} = \frac{(q^9; q^9)_\infty^3 (q^{18}; q^{18})_\infty^3}{(q^3; q^3)_\infty^5 (q^6; q^6)_\infty^3} \left( -2q^3 x(q^3) + \frac{1}{x^2(q^3)} \right), \quad (2.3)$$

$$\sum_{n=0}^{\infty} PD(3n+1) q^{3n+1} = q \frac{(q^9; q^9)_\infty^3 (q^{18}; q^{18})_\infty^3}{(q^3; q^3)_\infty^5 (q^6; q^6)_\infty^3} \left( 4q^3 x^2(q^3) + \frac{1}{x(q^3)} \right), \quad (2.4)$$

$$\sum_{n=0}^{\infty} PD(3n+2) q^{3n+2} = 3q^2 \frac{(q^9; q^9)_\infty^3 (q^{18}; q^{18})_\infty^3}{(q^3; q^3)_\infty^5 (q^6; q^6)_\infty^3}. \quad (2.5)$$

Thus Theorem 1.3 can be deduced from (2.5) by dividing both sides by  $q^2$  and substituting  $q^3$  by  $q$ . Similarly, Theorem 1.6 can be deduced from (2.4) by dividing both sides by  $q$  and substituting  $q^3$  by  $q$ . This completes the proof.  $\blacksquare$

It turns out that  $F(q^2)$  can be computed with the aid of some identities for cubic theta functions. These functions are introduced by Borwein, Borwein and Garvan [7] and are defined by

$$\begin{aligned} a(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \\ b(q) &= \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad \omega = e^{2\pi i/3}, \\ c(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}. \end{aligned}$$

Recall that

$$c(q) = 3 \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty}, \quad (2.6)$$

see Berndt, Bhargava and Garvan [6, Eq. (5.5)]. We shall also use the following identities for  $a(q)$  and  $c(q)$

$$a(q) = a(q^4) + 6q \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty (q^6; q^6)_\infty}, \quad (2.7)$$

$$c(q) = qc(q^4) + 3 \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty}, \quad (2.8)$$

$$a(q) = a(q^2) + 2q \frac{c^2(q^2)}{c(q)}. \quad (2.9)$$

Identity (2.7) for  $a(q)$  and identity (2.8) for  $c(q)$  are due to Hirschhorn, Garvan, and Borwein [9, Eqs.(1.36) and (1.34)]. Identity (2.9) for  $a(q)$  and  $c(q)$  is obtained by Berndt, Bhargava, Garvan [6, Eq. (6.3)].

We obtain the following identity on Ramanujan's cubic continued fraction.

**Theorem 2.2.** *Let*

$$x(q) = q^{-\frac{1}{3}}v(q) = \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty}.$$

*We have*

$$\frac{1}{x^2(q)} - 2qx(q) = 3q^2 \frac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^6}{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^6} + \frac{(q^4; q^4)_\infty^6}{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2 (q^{12}; q^{12})_\infty^2}. \quad (2.10)$$

*Proof.* We first establish a connection between Ramanujan's cubic continued fraction  $v(q)$  and the cubic theta function  $c(q)$ . It is easy to check that

$$\frac{1}{x^2(q)} = \frac{(q^3; q^6)_\infty^6}{(q; q^2)_\infty^2} = \frac{(q^2; q^2)_\infty^2}{(q^6; q^6)_\infty^6} \times \left( \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} \right)^2 = \frac{(q^2; q^2)_\infty^2}{9(q^6; q^6)_\infty^6} \times c^2(q), \quad (2.11)$$

$$2qx(q) = 2q \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty^3} = 2q \frac{(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty} \times \left( \frac{(q; q)_\infty}{(q^3; q^3)_\infty^3} \right) = 6q \frac{(q^6; q^6)_\infty^3}{(q^2; q^2)_\infty} \times \frac{1}{c(q)}. \quad (2.12)$$

We now consider the 2-dissection of  $1/x^2(q)$ . Identity (2.8) can be viewed as the 2-dissection of  $c(q)$ . Hence we deduce that

$$c^2(q) = \left( q^2 c^2(q^4) + 9 \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^4}{(q^2; q^2)_\infty^4 (q^{12}; q^{12})_\infty^2} \right) + q \left( 6c(q^4) \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} \right).$$

This yields the 2-dissection of  $1/x^2(q)$ ,

$$\begin{aligned}
\frac{1}{x^2(q)} &= \left( q^2 c^2(q^4) \frac{(q^2; q^2)_\infty^2}{9(q^6; q^6)_\infty^6} + \frac{(q^2; q^2)_\infty^2}{(q^6; q^6)_\infty^6} \frac{(q^4; q^4)_\infty^6 (q^6; q^6)_\infty^4}{(q^2; q^2)_\infty^4 (q^{12}; q^{12})_\infty^2} \right) \\
&\quad + q \left( 6c(q^4) \frac{(q^2; q^2)_\infty^2}{9(q^6; q^6)_\infty^6} \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} \right) \\
&= \left( q^2 \frac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^6}{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^6} + \frac{(q^4; q^4)_\infty^6}{(q^2; q^2)_\infty^2 (q^6; q^6)_\infty^2 (q^{12}; q^{12})_\infty^2} \right) \\
&\quad + 2q \left( \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^6; q^6)_\infty^4} \right). \tag{2.13}
\end{aligned}$$

Next, we aim to derive the 2-dissection of  $q/c(q)$ . By (2.9), we find

$$\frac{q}{c(q)} = \frac{a(q) - a(q^2)}{2c^2(q^2)}. \tag{2.14}$$

Substituting (2.7) into (2.14), we arrive at

$$\frac{q}{c(q)} = \frac{1}{2c^2(q^2)} \left( a(q^4) + 6q \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty (q^6; q^6)_\infty} - a(q^2) \right). \tag{2.15}$$

Using (2.9) with  $q$  replaced by  $q^2$ , we get

$$a(q^2) - a(q^4) = 2q^2 \frac{c^2(q^4)}{c(q^2)}.$$

Hence (2.15) can be written as

$$\begin{aligned}
\frac{q}{c(q)} &= \frac{1}{2c^2(q^2)} \left( -2q^2 \frac{c^2(q^4)}{c(q^2)} + 6q \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty (q^6; q^6)_\infty} \right) \\
&= -q^2 \frac{c^2(q^4)}{c^3(q^2)} + 3q \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{c^2(q^2) (q^2; q^2)_\infty (q^6; q^6)_\infty}.
\end{aligned}$$

Thus, we obtain the following 2-dissection of  $2qx(q)$ ,

$$\begin{aligned}
2qx(q) &= -6q^2 \frac{(q^6; q^6)_\infty^3 c^2(q^4)}{(q^2; q^2)_\infty c^3(q^2)} + 18q \frac{(q^6; q^6)_\infty^3 (q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{c^2(q^2) (q^2; q^2)_\infty^2 (q^6; q^6)_\infty} \\
&= -2q^2 \frac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^6}{(q^6; q^6)_\infty^6 (q^4; q^4)_\infty^2} + 2q \frac{(q^4; q^4)_\infty^2 (q^{12}; q^{12})_\infty^2}{(q^6; q^6)_\infty^4}. \tag{2.16}
\end{aligned}$$

Subtracting (2.16) from (2.13), we obtain (2.10). This completes the proof.  $\blacksquare$

*Proof of Theorem 1.5.* Substituting  $q^3$  with  $q$  in (2.3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} PD(3n)q^n &= \frac{(q^3; q^3)_{\infty}^3 (q^6; q^6)_{\infty}^3}{(q; q)_{\infty}^5 (q^2; q^2)_{\infty}^3} \left( -2qx(q) + \frac{1}{x^2(q)} \right) \\
&= \frac{(q^3; q^6)_{\infty}^3 (q^6; q^6)_{\infty}^6}{(q; q^2)_{\infty}^5 (q^2; q^2)_{\infty}^8} \left( -2qx(q) + \frac{1}{x^2(q)} \right) \\
&= \frac{1}{(q; q^6)_{\infty}^5 (q^3; q^6)_{\infty}^2 (q^5; q^6)_{\infty}^5} \times \frac{(q^6; q^6)_{\infty}^6}{(q^2; q^2)_{\infty}^8} \left( -2qx(q) + \frac{1}{x^2(q)} \right). \quad (2.17)
\end{aligned}$$

Applying (2.10) to (2.17), we are led to the generating function for  $PD(3n)$  in Theorem 1.5. This completes the proof.  $\blacksquare$

### 3 A combinatorial interpretation

In this section, we give a combinatorial interpretation of the congruence  $PD(3n+2) \equiv 0 \pmod{3}$ . In doing so, we introduce a rank for partitions with designated summands. We call this rank the *pd*-rank which enables us to divide the set of partitions of  $3n+2$  with designated summands into three equinumerous classes. The definition of the *pd*-rank is based on the following representation of a partition with designated summands by a pair of partitions.

**Theorem 3.1.** *There is a bijection  $\Delta$  between the set of partitions of  $n$  with designed summands and the set of pairs of partitions  $(\alpha, \beta)$  of  $n$ , where  $\alpha$  is an ordinary partition and  $\beta$  is a partition into parts  $\not\equiv \pm 1 \pmod{6}$ .*

To give a proof of the above theorem, we shall use the bijective proof of the following theorem of MacMahon given by Andrews, Eriksson, Petrov and Romik [4].

**Theorem 3.2.** ([11]) *The number of partitions of an integer  $n$  into parts  $\not\equiv \pm 1 \pmod{6}$  equals the number of partitions of  $n$  not containing any part exactly once.*

*Proof of Andrews, Eriksson, Petrov and Romik.* We construct a bijection  $\Phi$  from the set  $\mathcal{C}_n$  of partitions of  $n$  not containing any part exactly once to the set  $\mathcal{B}_n$  of partitions of  $n$  into parts not congruent to  $\pm 1 \pmod{6}$ . To describe the map  $\Phi$ , let  $\mu$  be a partition in  $\mathcal{C}_n$ . Write  $\mu$  as in the form of  $(1^{m_1} 2^{m_2} \dots l^{m_l})$ , where  $m_k$  is the multiplicity of  $k$  so that  $n = \sum_{k=1}^t k m_k$ . Since  $m_k \neq 1$  for any  $k$ , there is a unique way to write  $m_k$  as  $m_k = s_k + t_k$ , where  $s_k \in \{0, 3\}$  and  $t_k \in \{0, 2, 4, 6, 8, \dots\}$ . Now, the partition  $\lambda = \Phi(\mu) = (1^{b_1} 2^{b_2} \dots)$  is determined as follows:

$$\begin{aligned}
b_{6k+1} &= 0, & b_{6k+5} &= 0, \\
b_{6k+2} &= \frac{1}{2} t_{3k+1}, & b_{6k+4} &= \frac{1}{2} t_{3k+2}, \\
b_{6k+3} &= \frac{1}{3} s_{2k+1} + t_{6k+3}, & b_{6k+6} &= \frac{1}{3} s_{2k+2} + t_{6k+6}.
\end{aligned}$$

It is evident that  $\lambda$  is a partition into parts not congruent to  $\pm 1 \pmod 6$ . It is also apparent that one can recover the partition  $\mu$  from  $\lambda$  by reversing the above procedure. Hence  $\Phi$  is a bijection. This completes the proof.  $\blacksquare$

We are now in a position to present the proof of Theorem 3.1 by using the bijection  $\Phi$ .

*Proof of Theorem 3.1.* Let  $\lambda$  be a partition of  $n$  with designated summands. We wish to construct a pair of partitions  $(\alpha, \beta)$  of  $n$ , where  $\alpha$  is an ordinary partition and  $\beta$  is a partition into parts  $\not\equiv \pm 1 \pmod 6$ .

Suppose  $t$  is a magnitude that appears in  $\lambda$  and there are  $m_t$  parts equal to  $t$  among which the  $i$ -th part is designated. There are two cases.

- If  $i = 1$ , then move all the parts equal to  $t$  (including the designated part) in  $\lambda$  to the partition  $\alpha$ .
- If  $i \neq 1$ , then move  $i$  parts equal to  $t$  in  $\lambda$  to  $\gamma$  and  $(m_t - i)$  parts equal to  $t$  in  $\lambda$  to  $\alpha$ .

It can be seen that each part in  $\gamma$  occurs at least twice. Let  $\beta = \Phi(\gamma)$ . It is clear that  $\beta$  is a partition into parts  $\not\equiv \pm 1 \pmod 6$  and the above procedure can be reversed. Hence  $\Delta$  is a bijection. This completes the proof.  $\blacksquare$

The  $pd$ -rank of a partition  $\lambda$  with designated summands can be defined in terms of the pair of partitions  $(\alpha, \beta)$  under the map  $\Delta$ .

**Definition 3.3.** Let  $\lambda$  be a partition with designated summands and let  $(\alpha, \beta) = \Delta(\lambda)$ . Then the  $pd$ -rank of  $\lambda$ , denoted  $r_d(\lambda)$ , is defined by

$$r_d(\lambda) = l_e(\alpha) - l_e(\beta), \quad (3.1)$$

where  $l_e(\alpha)$  is the number of even parts of  $\alpha$  and  $l_e(\beta)$  is the number of even parts of  $\beta$ .

The following theorem shows that the  $pd$ -rank can be used to divide the set of partitions of  $3n + 2$  with designated summands into three equinumerous classes.

**Theorem 3.4.** For  $i = 0, 1, 2$ , let  $N_d(i, 3; n)$  denote the number of partitions of  $n$  with designated summands with  $pd$ -rank congruent to  $i \pmod 3$ . Then we have

$$N_d(0, 3; 3n + 2) = N_d(1, 3; 3n + 2) = N_d(2, 3; 3n + 2). \quad (3.2)$$

*Proof.* Let  $N_d(m; n)$  denote the number of partitions of  $n$  with designated summands with  $pd$ -rank  $m$ . By the definition of the  $pd$ -rank, we see that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_d(m; n) z^m q^n = \frac{1}{(zq^2; q^2)_{\infty} (q; q^2)_{\infty}} \times \frac{1}{(z^{-1}q^2; q^2)_{\infty} (q^3; q^6)_{\infty}}. \quad (3.3)$$



Setting  $z = \zeta = e^{\frac{2\pi i}{3}}$ , we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_d(m; n) \zeta^m q^n &= \sum_{n=0}^{\infty} \sum_{i=0}^2 N_d(i, 3; n) \zeta^i q^n \\
&= \frac{1}{(\zeta q^2; q^2)_{\infty} (q; q^2)_{\infty} (\zeta^{-1} q^2; q^2)_{\infty} (q^3; q^6)_{\infty}} \\
&= \frac{(-q^3; q^3)_{\infty}}{(q; q^2)_{\infty} (\zeta q^2; q^2)_{\infty} (\zeta^{-1} q^2; q^2)_{\infty}}. \tag{3.4}
\end{aligned}$$

Multiplying the right hand side of (3.4) by

$$\frac{(q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}},$$

and noting that

$$(1-x)(1-x\zeta)(1-x\zeta^2) = 1-x^3,$$

we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{i=0}^2 N_d(i, 3; n) \zeta^i q^n &= \frac{(-q^3; q^3)_{\infty}}{(q; q^2)_{\infty} (\zeta q^2; q^2)_{\infty} (\zeta^{-1} q^2; q^2)_{\infty}} \times \frac{(q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \\
&= \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \times \frac{(-q^3; q^3)_{\infty}}{(q^6; q^6)_{\infty}}.
\end{aligned}$$

By Gauss's identity [1, P. 23]

$$\frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}},$$

we get

$$\sum_{n=0}^{\infty} \sum_{i=0}^2 N_d(i, 3; n) \zeta^i q^n = \frac{(-q^3; q^3)_{\infty}}{(q^6; q^6)_{\infty}} \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}. \tag{3.5}$$

Since

$$\binom{n+1}{2} \equiv 0 \text{ or } 1 \pmod{3},$$

the coefficient of  $q^{3n+2}$  in (3.5) is zero. It follows that

$$N_d(0, 3; 3n+2) + N_d(1, 3; 3n+2)\zeta + N_d(2, 3; 3n+2)\zeta^2 = 0.$$

Since  $1 + \zeta + \zeta^2$  is the minimal polynomial in  $\mathbb{Z}[\zeta]$ , we conclude that

$$N_d(0, 3; 3n+2) = N_d(1, 3; 3n+2) = N_d(2, 3; 3n+2).$$

This completes the proof. ■

For example, for  $n = 5$ , we have  $PD(5) = 15$ . The fifteen partitions of 5 with designated summands, the corresponding pairs of partitions, along with the  $pd$ -ranks modulo 3 are listed in Table 3.1. It can be checked that

$$N_d(0, 3; 5) = N_d(1, 3; 5) = N_d(2, 3; 5) = 5.$$

$\lambda$	$(\alpha, \beta) = \Delta(\lambda)$	$r_d(\lambda) \pmod{3}$
$5'$	$(5, \emptyset)$	0
$4' + 1'$	$(4 + 1, \emptyset)$	1
$3' + 2'$	$(3 + 2, \emptyset)$	1
$3' + 1' + 1$	$(3 + 1 + 1, \emptyset)$	0
$3' + 1 + 1'$	$(3, 2)$	2
$2' + 2 + 1'$	$(2 + 2 + 1, \emptyset)$	2
$2 + 2' + 1'$	$(1, 4)$	2
$2' + 1' + 1 + 1$	$(2 + 1 + 1 + 1, \emptyset)$	1
$2' + 1 + 1' + 1$	$(2 + 1, 2)$	0
$2' + 1 + 1 + 1'$	$(2, 3)$	1
$1' + 1 + 1 + 1 + 1$	$(1 + 1 + 1 + 1 + 1, \emptyset)$	0
$1 + 1' + 1 + 1 + 1$	$(1 + 1 + 1, 2)$	2
$1 + 1 + 1' + 1 + 1$	$(1 + 1, 3)$	0
$1 + 1 + 1 + 1' + 1$	$(1, 2 + 2)$	1
$1 + 1 + 1 + 1 + 1'$	$(\emptyset, 3 + 2)$	2

Table 3.1: The case for  $n = 5$ .

**Acknowledgments.** This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

## References

- [1] G.E. Andrews, The Theory of Partitions, Encycl. Math. and Its Applications, Vol. 2, Addison-Wesley, Reading, 1976.
- [2] G.E. Andrews, R.P. Lewis, and J. Lovejoy, Partitions with designated summands, Acta Arith. 105 (2002) 51–66.
- [3] G.E. Andrews and B.C. Berndt, Ramanujan’s Lost Notebook. Part I, Springer, New York, 2005.
- [4] G. Andrews, H. Eriksson, F. Petrov, and D. Romik, Integrals, partitions and MacMahon’s theorem, J. Comb. Theory A 114 (2007) 545–554.

- [5] G.E. Andrews and S.C.F. Rose, MacMahon's sum-of-divisors functions, Chebyshev polynomials, and quasi-modular forms, *J. Reine Angew. Math.*, to appear.
- [6] B.C. Berndt, S. Bhargava, and F.G. Garvan, Ramanujan's theories of elliptic functions to alternative bases, *Trans. Amer. Math. Soc.* 347 (1995) 4163–4244.
- [7] J.M. Borwein, P.B. Borwein, and F.G. Garvan, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.* 343 (1994) 35–47.
- [8] H.-C. Chan, Ramanujan's cubic continued fraction and a generalization of his “most beautiful identity”, *Int. J. Number Theory* 6 (2010) 673–680.
- [9] M.D. Hirschhorn, F. Garvan, and J. Borwein, Cubic analogues of the Jacobian theta function  $(z, q)$ , *Canad. J. Math.* 45 (1993) 673–694.
- [10] P. A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, *Proc. London Math. Soc. Ser. 2* 19 (1919) 75–113.
- [11] P.A. MacMahon, *Combinatory Analysis*, vols. I and II, Cambridge Univ. Press, Cambridge, 1915–1916, reissued, Chelsea, 1960.
- [12] J. Sturm, On the congruence properties of modular forms, *Springer Lect. Notes in Math.* Vol.1240, pp. 275–280, Springer-Verlag, Berlin/New York, 1984.